On the flow of a conducting fluid past a magnetized sphere

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In the steady flow of an incompressible, inviscid, conducting fluid past a magnetized sphere, the first-order effects of the magnetic field and the conductivity are studied. Paraboloidal wakes of vorticity and magnetic intensity are formed, the former being half the size of the latter. The vorticity, generated by the non-conservative electromagnetic force, is logarithmically infinite on the sphere. For the case of a dipole of moment M at the centre of a sphere of radius a, the drag coefficient is

$$C_D = \frac{144\mu'^2}{5(2\mu+\mu')^2}\beta R_M,$$

where μ and μ' are the permeabilities of the fluid and sphere, respectively, β is the ratio of the representative magnetic pressure $\mu M^2/2a^6$ to the free-stream dynamic pressure, and R_M is the magnetic Reynolds number.

1. Introduction

So far in the study of the effects of a magnetic field on the flow of a conducting fluid past a finite body, attention has been focused on problems in which the magnetic field is applied externally. It is of some interest to consider a case in which the magnetic field originates in the body itself. In all such questions the generation of vorticity by the action of the non-conservative electromagnetic force is of special concern.

Here we consider the steady flow of an incompressible, inviscid fluid past a sphere of arbitrary conductivity, in which there is an arbitrary, axially symmetric, magnetic distribution. For such a configuration the current lines are circles. The results are described for a general distribution and are evaluated explicitly for a dipole situated at the centre.

Our investigation is concerned with the first-order effects of the magnetic field and the conductivity of the fluid. This leads to a regular perturbation in β, the ratio of a representative magnetic pressure to the free-stream dynamic pressure. A similar perturbation in the magnetic Reynolds number R_M is not uniformly valid; the perturbation being singular at infinity.

The situation is similar to that in Oseen's approximation to the viscous flow past a sphere: wakes (of magnetic intensity and vorticity) are formed whose

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boundaries are paraboloids with linear dimensions proportional to $1/R_M$. The vorticity wake is half the size of the magnetic wake.

The first term in the β-expansion for the magnetic field can be obtained as a regular perturbation in R_M after an exponential factor (representing the non-uniformity at infinity) has been extracted. Once this is determined, the first term in the β-perturbation for the vorticity can be evaluated by integration along the streamlines of the potential flow past the sphere, using the fact that the vorticity vanishes at infinity upstream. Now the time during which a fluid element is under the influence of the non-conservative force, in its passage through the neighbourhood of the front stagnation point, tends to infinity logarithmically with its closest approach to that point. Consequently, the vorticity becomes logarithmically infinite on the sphere.

Drag on the sphere derives from two sources, the fluid pressure and the Maxwell stress. For the dipole case there is no contribution from the pressure, and the drag coefficient is $[144\mu'^2/5(2\mu+\mu')^2]\beta R_M$, where μ and μ' are the permeabilities of the fluid and sphere, respectively.*

The flow is completely independent of the conductivity of the sphere, provided it is finite. On the other hand, if it is assumed that the field was originally frozen into the sphere, the results are different. This case is discussed in the last section.

2. Basic equations

The equations governing the steady motion of an incompressible, inviscid, electrically conducting fluid of constant properties are:

$$\left.\begin{array}{l}(a)\ \operatorname{div}\mathbf{v}=0,\quad (b)\ \operatorname{curl}\mathbf{v}\times\mathbf{v}=-\operatorname{grad}[(p/\rho_0)+\tfrac{1}{2}v^2]+(\mu/\rho_0)\operatorname{curl}\mathbf{H}\times\mathbf{H},\\ (c)\ \operatorname{curl}\mathbf{H}=\sigma(\mathbf{E}+\mu\mathbf{v}\times\mathbf{H}),\quad (d)\ \operatorname{curl}\mathbf{E}=0,\quad (e)\ \operatorname{div}\mathbf{H}=0.\end{array}\right\} \quad (1)$$

For an axially symmetric motion in which the velocity $\mathbf{v}$ and magnetic intensity $\mathbf{H}$ lie in the meridian plane and are independent of the azimuthal angle, the conduction equation (1*c*) shows that the electric field $\mathbf{E}$ is perpendicular to this plane and also independent of the azimuthal angle. It then follows from (1*d*) that $\mathbf{E}=0$, if it is to be finite at the axis. When $\mathbf{v}$, $\mathbf{r}$ and $\mathbf{H}$ are now made dimensionless by referring them to the velocity at infinity U, the radius of the sphere a, and a representative magnetic intensity h, respectively, the equations reduce to

$$\left.\begin{array}{l}(a)\ \operatorname{div}\mathbf{v}=0,\quad (b)\operatorname{curl}\mathbf{v}\times\mathbf{v}=-\operatorname{grad}P+\beta\operatorname{curl}\mathbf{H}\times\mathbf{H},\\ (c)\ \operatorname{curl}\mathbf{H}=R_M\mathbf{v}\times\mathbf{H},\quad (d)\ \operatorname{div}\mathbf{H}=0,\end{array}\right\} \quad (2)$$

where
$$P=p+\tfrac{1}{2}v^2,\quad \beta=\frac{\mu h^2}{\rho_0 U^2},\quad R_M=Ua\mu\sigma,$$

and p is now the pressure divided by $\rho_0 U^2$.

When there is no magnetic field, β is zero and (2*b*) shows that P is constant along streamlines. From the assumed uniform conditions at infinity it then follows that P is constant throughout and hence [using (2*b*) again] that $\operatorname{curl}\mathbf{v}=0$. In conjunction with (2*a*) this yields the potential solution

$$\mathbf{v}=\mathbf{v}_0=\left\{\left(1-\frac{1}{r^3}\right)\cos\theta,\ -\left(1+\frac{1}{2r^3}\right)\sin\theta,\ 0\right\}, \quad (3)$$

* The total drag is more easily obtained from the Joule dissipation (Chopra & Singer 1958).

satisfying the condition of zero normal velocity at the sphere $r = 1$ and tending to the given uniform stream $(\cos\theta, -\sin\theta, 0)$ at infinity. Here we use spherical polar co-ordinates with $\theta = 0$ pointing downstream (see figure 1). The corresponding pressure field $p = p_0$ is then determined by the constancy of P.

For weak magnetic fields (small h) we expand in powers of β

$$\mathbf{v} = \mathbf{v}_0 + \beta\mathbf{v}_1 + \ldots, \quad p = p_0 + \beta p_1 + \ldots, \quad \mathbf{H} = \mathbf{H}_0 + \beta\mathbf{H}_1 + \ldots.$$

Note that $\mathbf{H}_0$ is not zero. To obtain the magnetic field, $\mathbf{H}$ must be multiplied by h and it is the latter which tends to zero in the limit. According to (2c) and (2d), $\mathbf{H}_0$ must satisfy

$$\operatorname{curl}\mathbf{H}_0 = R_M \mathbf{v}_0 \times \mathbf{H}_0, \quad \operatorname{div}\mathbf{H}_0 = 0, \tag{4}$$

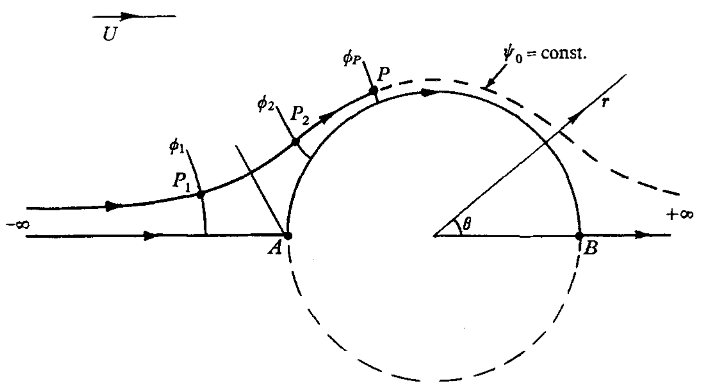

FIGURE 1. Integration along the potential flow streamlines.

where $\mathbf{v}_0$ is given by (3). The remaining equations (2a) and (2b) give

$$\operatorname{curl}\mathbf{v}_1 \times \mathbf{v}_0 = -\operatorname{grad} P_1 + \operatorname{curl}\mathbf{H}_0 \times \mathbf{H}_0, \quad \operatorname{div}\mathbf{v}_1 = 0, \tag{5}$$

and these determine $\mathbf{v}_1$ and p_1 once $\mathbf{H}_0$ is known; here

$$P_1 = p_1 + \mathbf{v}_0 \cdot \mathbf{v}_1$$

and we have used the fact that $\operatorname{curl}\mathbf{v}_0 = 0$.

We shall restrict the discussion to $\mathbf{H}_0$, $\mathbf{v}_1$, and p_1. It is easily seen, however, that at each stage in the approximation the terms in $\mathbf{v}$ and p are determined before the term in $\mathbf{H}$.

3. Determination of $\mathbf{H}_0$

The subscripts 0 on $\mathbf{H}_0$ and 1 on $\mathbf{v}_1$ and p_1 will now be dropped. In axially symmetric flow the second of equations (4) allows us to write for $\mathbf{H} = (H_r, H_\theta, 0)$

$$H_r = \frac{1}{r^2\sin\theta}\frac{\partial A}{\partial\theta}, \quad H_\theta = -\frac{1}{r\sin\theta}\frac{\partial A}{\partial r}, \tag{6}$$

where A is a function of r and θ alone. On introducing this into the single non-zero component of the first equation, we find that A must satisfy

$$\frac{\partial^2 A}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial A}{\partial\theta}\right) = R_M\left[\left(1-\frac{1}{r^3}\right)\cos\theta\frac{\partial A}{\partial r} - \frac{1}{r}\left(1+\frac{1}{2r^3}\right)\sin\theta\frac{\partial A}{\partial\theta}\right]. \tag{7}$$

For small values of R_M an appropriate solution of this equation may be found by a perturbation expansion. The perturbation is not regular at infinity, however, since the right-hand side of (7) vanishes more slowly than the left, as $r \to \infty$, when A is algebraic. In fact, if small terms in r on the right-hand side are neglected, the equation becomes*

$$\frac{\partial^2 A}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial A}{\partial\theta}\right) = R_M\left[\cos\theta\frac{\partial A}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial A}{\partial\theta}\right],$$

of which simple solutions, for arbitrary R_M, are

$$A = r^{\frac{1}{2}}K_{n+\frac{1}{2}}(\tfrac{1}{2}R_M r)\exp(\tfrac{1}{2}R_M r\cos\theta)\sin^2\theta\, P_n'(\cos\theta). \tag{8}$$

Here n is a positive integer, K is a Bessel function, and P the Legendre polynomial. Apart from the exponential, A is a product solution, chosen so as to be regular on the axis and finite at infinity. For large values of r,

$$A \sim \surd(\pi/R_M)\exp[-\tfrac{1}{2}R_M r(1-\cos\theta)]\sin^2\theta\, P_n'(\cos\theta),$$

which gives an exponential decrease for all $\theta \neq 0$. On the other hand, for r fixed, $R_M^{n+\frac{1}{2}}A$ tends to a multiple of $\sin^2\theta\, P_n'(\cos\theta)/r^n$ as $R_M \to 0$; that is, a solution of (7) with $R_M = 0$ (curl $\mathbf{H} = 0$).

Thus the magnetic field is swept into a wake behind the sphere, whose boundary is the paraboloid of revolution $r(1-\cos\theta) = c/R_M$ (c a suitably chosen constant). As $R_M \to 0$ the wake broadens so as to fill the whole of space in the limit.

It is now clear how the perturbation must be carried out. In equation (7) we set

$$A = \exp[-\tfrac{1}{2}\rho(1-\cos\theta)]\alpha \quad (\rho = R_M r),$$

and obtain for α the equation

$$\frac{\partial^2\alpha}{\partial\rho^2} - \frac{\partial\alpha}{\partial\rho} + \frac{\sin\theta}{\rho^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\alpha}{\partial\theta}\right)$$
$$= -R_M^3\left[\frac{\cos\theta}{\rho^3}\frac{\partial\alpha}{\partial\rho} + \frac{\sin\theta}{2\rho^4}\frac{\partial\alpha}{\partial\theta} - (1-\cos\theta)(1+3\cos\theta)\frac{1}{4\rho^3}\alpha\right]. \tag{9}$$

With $R_M = 0$ this has the product solutions

$$\alpha = \rho^{\frac{1}{2}}e^{\rho/2}K_{n+\frac{1}{2}}(\tfrac{1}{2}\rho)\sin^2\theta P_n'(\cos\theta), \tag{10}$$

in accordance with (8). We now take a suitable combination of these solutions (so as to satisfy the boundary conditions at the sphere) and use it to generate the solution of (9) by successive approximation, ensuring at each stage that the boundary conditions are satisfied (to the appropriate order in R_M) by adding a suitable combination of (10).

The factor R_M^3 on the right-hand side of (9) does not imply that the perturbation is $O(R_M^3)$. On replacing ρ by $R_M r$, it becomes R_M; and at any finite r the approximations proceed in powers of R_M. However, there is an immediate exception due

* In Oseen's approximation to the viscous flow past a sphere, this is the equation satisfied by the vorticity, which is hardly surprising since the same physical processes are involved and the same type of approximation has been made [$\mathbf{v}_0$ replaced by the velocity at infinity in (4)].

to the fact that the disturbance part of $\mathbf{v}_0$ is a dipole field: the first two terms on the right in (9) cancel when the leading term in the first approximation is substituted.

The correct combination of (10) to take when there is a dipole at the centre of the sphere is

$$\alpha = \kappa R_M\left(\frac{1}{\rho}+\frac{1}{2}\right)\sin^2\theta + \lambda R_M^3\left(\frac{1}{\rho^2}+\frac{1}{2\rho}+\frac{1}{12}\right)\sin^2\theta\cos\theta,$$

containing the solutions $n = 1$ and $n = 2$.* Here κ and λ are constants which must be determined in conjunction with the field inside the sphere. Since the first perturbation, found by substituting this expression into the right-hand side of (9) and integrating, provides corrections $O(R_M^2)$, we may neglect the uncorrected terms at this stage and set

$$A = \exp\left[-\tfrac{1}{2}R_M r(1-\cos\theta)\right]\{(\kappa/r)\sin^2\theta + R_M[\tfrac{1}{2}\kappa + (\lambda/r^2)\cos\theta]\sin^2\theta\}. \quad (11)$$

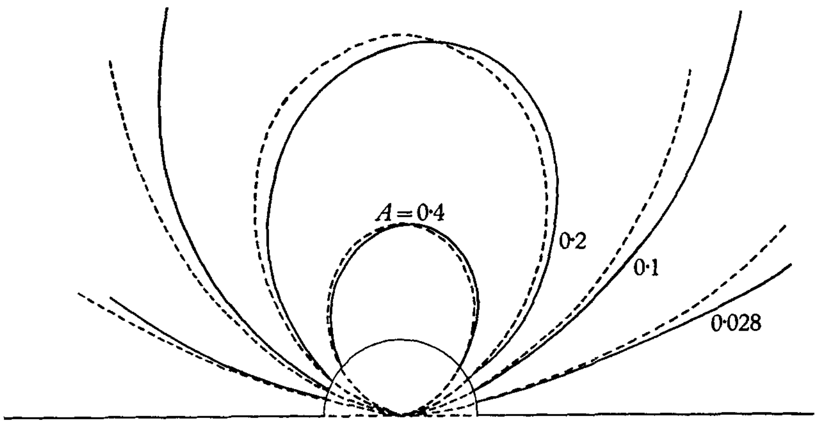

FIGURE 2. Distortion of the magnetic field of a dipole, for $R_M = 0{\cdot}1$ and $\mu' = \mu$. Broken lines show undisturbed field. Inside the sphere ($r = 1$) the fields are indistinguishable.

Inside the sphere we again have $\mathbf{E} = 0$, so that $\operatorname{curl}\mathbf{H} = \sigma'\mathbf{E} = 0$, whatever the conductivity σ' of the sphere. Hence we take

$$A = (1/r)\sin^2\theta + (kr^2 + R_M l r^3\cos\theta)\sin^2\theta, \quad (12)$$

the first term being due to the dipole and the rest representing the (regular) irrotational disturbance field (k and l are constants to be determined). We can now identify h as M/a^3, where M is the moment of the dipole.

The validity of (11) and (12) now follows from the fact that κ, λ, k and l can be chosen so that the boundary conditions on the sphere are satisfied to order R_M. Thus the continuity of the tangential component of $\mathbf{H}$ requires

$$1-2k = \kappa, \quad 3l = -2\lambda,$$

and the continuity of the normal component of magnetic induction requires

$$\mu'(1+k) = \mu\kappa, \quad \mu' l = \mu(\tfrac{1}{2}\kappa+\lambda),$$

* $K_{n+\frac{1}{2}}(x)$ is a polynomial of degree n in $1/x$ multiplied by $e^{-x}/x^{\frac{1}{2}}$.

where μ' is the permeability of the sphere. These are the only conditions; and they lead to the values

$$k = \frac{\mu-\mu'}{2\mu+\mu'}, \quad l = \frac{3\mu\mu'}{(2\mu+\mu')(3\mu+2\mu')}, \quad \kappa = \frac{3\mu'}{2\mu+\mu'}, \quad \lambda = -\frac{9\mu\mu'}{2(2\mu+\mu')(3\mu+2\mu')}.$$

We shall not take the approximation any further than (11) and (12), though it may easily be done. From (6) we find for the field outside the sphere

$$\left.\begin{aligned} H_r &= \exp\left[-\tfrac{1}{2}R_M r(1-\cos\theta)\right]\left\{\frac{2\kappa}{r^3}(1+\tfrac{1}{2}R_M r)\cos\theta + \frac{R_M}{r^4}\left[(\tfrac{1}{2}\kappa r^2+3\lambda)\cos^2\theta-(\tfrac{1}{2}\kappa r^2+\lambda)\right]\right\}, \\ H_\theta &= \exp\left[-\tfrac{1}{2}R_M r(1-\cos\theta)\right]\left\{\frac{\kappa}{r_3}(1+\tfrac{1}{2}R_M r)\sin\theta - \frac{R_M}{r^4}(\tfrac{1}{2}\kappa r^2-2\lambda)\sin\theta\cos\theta\right\}, \end{aligned}\right\} \quad (13)$$

to the same degree of approximation (see figure 2).

4. Determination and behaviour of the vorticity

The vorticity vector curl $\mathbf{v}$ is perpendicular to the meridian plane, and hence has only one component, ω. According to (5), it satisfies the equations

$$\omega\left(1+\frac{1}{2r^3}\right)\sin\theta = -\frac{\partial P}{\partial r}+F_r, \quad (14a)$$

$$\omega\left(1-\frac{1}{r^3}\right)\cos\theta = -\frac{1}{r}\frac{\partial P}{\partial\theta}+F_\theta, \quad (14b)$$

where $F = (F_r, F_\theta, 0) = \text{curl}\,\mathbf{H}\times\mathbf{H} = R_M(\mathbf{v}_0\times\mathbf{H})\times\mathbf{H}$. From (3) and (13) we find

$$\begin{aligned} F_r = {} & \exp\left[-R_M r(1-\cos\theta)\right]\{-(3\kappa^2 R_M/r^6)(1+R_M r)\sin^2\theta\cos\theta \\ & + (\kappa R_M^2/r^{10})\sin^2\theta[(\tfrac{3}{2}\kappa r^5-11\lambda r^3-\tfrac{3}{4}\kappa r^2+\tfrac{1}{2}\lambda)\cos^2\theta+(2r^3+1)(\tfrac{1}{4}\kappa r^2+\tfrac{1}{2}\lambda)]\}, \end{aligned} \quad (15a)$$

$$\begin{aligned} F_\theta = {} & \exp\left[-R_M r(1-\cos\theta)\right]\{(6\kappa^2 R_M/r^6)(1+R_M r)\sin\theta\cos^2\theta \\ & + (\kappa R_M^2/r^{10})\sin\theta\cos\theta[(\tfrac{3}{2}\kappa r^5+19\lambda r^3+\tfrac{3}{2}\kappa r^2-\lambda)\cos^2\theta-(5r^3+1)(\tfrac{1}{2}\kappa r^2+\lambda)]\}, \end{aligned} \quad (15b)$$

so that on eliminating P from equations (14), we obtain

$$\left(r-\frac{1}{r^2}\right)\cos\theta\frac{\partial\tilde{\omega}}{\partial r}-\left(1+\frac{1}{2r^3}\right)\sin\theta\frac{\partial\tilde{\omega}}{\partial\theta} = f(r,\theta;\,R_M), \quad (16)$$

where $\tilde{\omega} = \omega/(r\sin\theta)$ and

$$\begin{aligned} f(r,\theta;R_M) &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial r}(rF_\theta)-\frac{\partial}{\partial\theta}(F_r)\right] \\ &= \exp\left[-R_M r(1-\cos\theta)\right]\left\{-\frac{3\kappa^2 R_M}{r^7}(1+R_M r)(7\cos^2\theta+1)\right. \\ &\quad \left.+\frac{\kappa R_M^2}{r^{11}}\cos\theta[(-3\kappa r^5-70\lambda r^3-\tfrac{15}{2}\kappa r^2+7\lambda)\cos^2\theta+(9\kappa r^5+6\lambda r^3+\tfrac{3}{2}\kappa r^2+9\lambda)]\right\}. \end{aligned}$$

Note that the argument of the exponential has been doubled, which means that the size of the vorticity wake is only half that of the magnetic field.

The left-hand side of (16) represents a differentiation along the streamlines of the potential flow (3). Hence it is convenient to introduce the potential function and stream function of this flow

$$\phi_0 = r\left(1+\frac{1}{2r^3}\right)\cos\theta, \quad \psi_0 = r\sqrt{\{1-(1/r^3)\}}\sin\theta.*$$

When these are taken as new independent variables in place of r and θ, the $\tilde{\omega}$-equation becomes

$$r\left[\left(1-\frac{1}{r^3}\right)^2\cos^2\theta+\left(1+\frac{1}{2r^3}\right)^2\sin^2\theta\right]\frac{\partial\tilde{\omega}}{\partial\phi_0} = f(r,\theta;R_M).$$

Thus we find that at any point P

$$\tilde{\omega} = \int_{-\infty}^{\phi_P}\frac{r^5 f(r,\theta;R_M)}{[(r^3-1)^2\cos^2\theta+(r^3+\frac{1}{2})^2\sin^2\theta]}d\phi_0, \tag{17}$$

where the integrand is to be expressed in terms of ϕ_0 and ψ_0, and the integration taken with ψ_0 fixed. The choice for the lower limit ensures that on each streamline $\omega\to 0$ as $r\to\infty$ upstream. This is the only boundary condition needed to determine the vorticity.

Of special interest is the behaviour of ω at the sphere. Note that the integrand of $\tilde{\omega}$ becomes infinite at A (figure 1), through which passes the dividing streamline connecting $-\infty$ to any point on the sphere. Hence we consider the integration over a short segment P_1P_2 of a neighbouring streamline (ψ_0 small), and let ψ_0 tend to zero (keeping the limits $\phi_0 = \phi_1$, ϕ_2 fixed). Now, for $\psi_0 = \text{const.}$,

$$d\phi_0 = [(r^3-1)^2\cos^2\theta+(r^3+\tfrac{1}{2})^2\sin^2\theta]\frac{dr}{r^3(r^3-1)\cos\theta},$$

so that the dominant part of the integral is

$$I = \int_{\phi_0=\phi_1}^{\phi_0=\phi_2}\left[\frac{f(1,\pi;R_M)}{-3(r-1)}\right]dr = \tfrac{1}{3}f(1,\pi;R_M)\log\left(\frac{r_1-1}{r_2-1}\right),$$

the remainder tending to a finite limit as $\psi_0\to 0$. Since the integrals taken over the ranges $(-\infty,\phi_1)$ and (ϕ_2,ϕ_P) also tend to finite limits, this is the leading part of the complete integral (17)†. Finally, we note that r_2-1 may be replaced by r_P-1 in I, since P_2 and P lie on the same streamline. It follows that, if terms which tend to finite limits are omitted,

$$\begin{aligned}\omega &= -\tfrac{1}{3}\sin\theta_P f(1,\pi;R_M)\log(r_P-1)\\ &= 8R_M[\kappa^2-\kappa(\kappa+2\lambda)R_M+O(R_M^2)]\sin\theta_P\log(r_P-1),\end{aligned} \tag{18}$$

when P is close to the sphere.

The vorticity is logarithmically infinite only on the sphere. For a point P close to the axis downstream, a term $\log(r_2-1)$ still arises from integration near A (and a similar term from integration near the rear stagnation point B), but this time it is replaced by $\log(\sin^2\theta_P)$, a quantity which, when multiplied by $\sin\theta_P$, tends to zero with θ_P. It is easily seen that ω is also zero on the axis upstream.

* This is the square root of what is usually taken to be the stream function.

† For more details, see Appendix 1.

It is also of interest to see how the vorticity dies off at large distances. More precisely, we shall assume that $\xi_P = R_M r_P(1-\cos\theta_P)$ is large: in particular, this ensures that r is large at every upstream point on the streamline through P, so that the integrand in (17) may be replaced by

$$-\frac{3\kappa^2 R_M^2}{r^7}(\cos^3\theta + 7\cos^2\theta - 3\cos\theta + 1)\exp[-R_M r(1-\cos\theta)],$$

while, effectively, $\qquad \phi_0 = r\cos\theta, \quad \psi_0 = r\sin\theta.$

If now we change the integration variable to

$$u = \frac{r(1-\cos\theta)}{r_P(1-\cos\theta_P)} - 1,$$

the integral may be written as

$$\tilde{\omega} = \frac{\kappa^2 R_M^8}{\xi_P^6} e^{-\xi_P}\int_0^\infty e^{-\xi_P u} g(u, \cos\theta_P)\, du,$$

where g is a rational function of u with

$$g(0, \cos\theta_P) = -3(1-\cos\theta_P)^6(\cos^3\theta_P + 7\cos^2\theta_P - 3\cos\theta_P + 1)$$

and poles at $u = -1 \pm i\cot(\frac{1}{2}\theta_P)$. It now follows from Watson's lemma [see Copson 1935, p. 218] on asymptotic expansions that, for each fixed θ_P,

$$\omega \sim -\frac{3\kappa^2 R_M^2}{r_P^5}\sin\theta_P(\cos^3\theta_P + 7\cos^2\theta_P - 3\cos\theta_P + 1) \times \exp[-R_M r_P(1-\cos\theta_P)]\left\{1 + O\left(\frac{1}{\xi_P}\right)\right\}$$

when ξ_P is large.

5. Expansion of ω

The velocity field corresponding to ω can be written down by using the known velocity field of a circular vortex in the presence of a fixed sphere (Yeh, Martinek & Ludford 1955). The resulting formula is too complicated for our purposes, and accordingly we resort to a series expansion for ω, which, although it can in principle be derived from (17), will in fact be obtained directly from (16). It turns out that only certain (Fourier) components of the velocity field can be so determined, but fortunately the only component needed for the computation of the drag is one which can.

The stream function ψ corresponding to ω satisfies the equation

$$\frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = -r\sin\theta\,\omega, \tag{19}$$

and in terms of it the velocity components are

$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{20}$$

Since the product solutions of (19) for $\omega = 0$ which are regular on the axis have the form

$$\psi = \left(A_n r^{n+1} + \frac{B_n}{r^n}\right) \sin^2\theta P'_n(\cos\theta) \quad (n \text{ a positive integer}), \tag{21}$$

it is clear that we should expand in the form

$$\omega = \sum_{n=1}^{\infty} \omega_n(r) \sin\theta P'_n(\cos\theta). \tag{22}$$

Correspondingly the right-hand side of (16) is written

$$f(r,\theta;\, R_M) = \sum_{n=1}^{\infty} \Omega_n(r)\, P'_n(\cos\theta), \tag{23}$$

where the Ω_n are determined by suitable expansions of F_r and F_θ (see next section).

When the series (22) is substituted into the left-hand side of (16) and the result brought into the same form as the right-hand side (23), we find the recurrence relation

$$\frac{n+2}{2n+3}\left[\left(r-\frac{1}{r^2}\right)\omega'_{n+1} + \left(n+2+\frac{n+5}{2r^3}\right)\omega_{n+1}\right] + \frac{n-1}{2n-1}\left[\left(r-\frac{1}{r^2}\right)\omega'_{n-1} - \left(n-1+\frac{n-4}{2r^3}\right)\omega_{n-1}\right] = r\Omega_n(r),$$

on equating coefficients of $P'_n(\cos\theta)$. Its integrated form is

$$\frac{n+2}{2n+3}\frac{(r^3-1)^{\frac{1}{2}(n+3)}}{r^{\frac{1}{2}(n+5)}}\,\omega_{n+1} = \int_1^r \left(\frac{\alpha^3-1}{\alpha}\right)^{\frac{1}{2}(n+1)} \left\{\alpha\Omega_n(\alpha) - \frac{n-1}{2n-1}\left[\left(\alpha-\frac{1}{\alpha^2}\right)\omega'_{n-1} - \left(n-1+\frac{n-4}{2\alpha^3}\right)\omega_{n-1}\right]\right\} d\alpha.$$

The lower limit of integration is determined by the requirement that ω_n should not become algebraically infinite on $r = 1$.

For $n = 1$ we obtain

$$\omega_2 = \frac{5}{3}\frac{r^3}{(r^3-1)^2}\int_1^r (\alpha^3-1)\,\Omega_1(\alpha)\,d\alpha, \tag{24}$$

and then $\omega_4, \omega_6, \omega_8, \ldots$ follow successively. On the other hand, for $n = 2$ the relation involves both ω_1 and ω_3, and ω_1 must be chosen before ω_3, ω_5, $\omega_9, \ldots$ can be determined. This must be done in such a way that the complete series (22) tends to zero at infinitely large distances upstream, and, presumably, ω_1 is fixed by this condition. In any event, in order to obtain the drag on the sphere it is sufficient to know ω_2.

The $\psi_n(r)$ in the expansion

$$\psi = \sum_{n=1}^{\infty} \psi_n(r) \sin^2\theta\, P'_n(\cos\theta)$$

of the stream function is determined, according to (19) and (22), by the equation

$$\psi''_n - \frac{n(n+1)}{r^2}\psi_n = -r\omega_n.$$

The solutions which vanish on $r = 1$ are given by

$$\psi_n = \frac{1}{2n+1}\int_1^r \left(\frac{\xi^{n+2}}{r^n} - \frac{r^{n+1}}{\xi^{n-1}}\right)\omega_n(\xi)\,d\xi + A_n\left(r^{n+1} - \frac{1}{r^n}\right), \tag{25}$$

and we choose
$$A_n = \frac{1}{2n+1}\int_1^\infty \frac{\omega_n(\xi)}{\xi^{n-1}}\,d\xi, \tag{26}$$

so that ψ_n will remain finite at infinity.

Once the ψ_n have been determined, the velocity field follows from (20). In particular,

$$v_\theta = \sum_{n=1}^{\infty} v_n(r)\sin\theta P_n'(\cos\theta), \quad v_n(r) = -\psi_n'(r)/r.$$

The only detailed piece of information we need concerning this series is the value of v_2 on $r = 1$ (see §6). From (25) and (26) we find

$$\begin{aligned} v_2(1) &= -\psi_n'(1) = -5A_2 \\ &= -\frac{5}{3}\int_1^\infty \frac{\xi^2}{(\xi^3-1)^2}\,d\xi\int_1^\xi (\alpha^3-1)\,\Omega_1(\alpha)\,d\alpha \\ &= -\frac{5}{9}\int_1^\infty \Omega_1(\alpha)\,d\alpha. \end{aligned} \tag{27}$$

6. The drag

The force exerted by the fluid on the sphere has two parts, one due to the fluid pressure and the other to the Maxwell stress. We consider these in turn.

Assume that the F_θ appearing in equation (14*b*) has been expanded in the form

$$F_\theta = \sum_{n=1}^{\infty} a_n(r)\sin\theta P_n'(\cos\theta). \tag{28}$$

Then on the sphere we find by integration of (14*b*) that

$$P = p + \mathbf{v}_0.\mathbf{v} = -\sum_{n=0}^{\infty} a_n(1)\,P_n(\cos\theta),$$

where $a_0(1)$ is an undetermined constant of integration. Hence the total drag force due to pressure is $\rho_0 U^2 a^2 D_p$ where

$$\begin{aligned} D_p &= -\beta\int_0^\pi p\cos\theta\,.\,2\pi\sin\theta\,d\theta \\ &= 2\pi\beta\int_0^\pi\left[-\tfrac{3}{2}\sin\theta\sum_{n=1}^{\infty} v_n(1)\sin\theta P_n'(\cos\theta) + \sum_{n=0}^{\infty} a_n(1)\,P_n(\cos\theta)\right]\sin\theta\,\cos\theta\,d\theta \\ &= 2\pi\beta[-\tfrac{6}{5}v_2(1) + \tfrac{2}{3}a_1(1)]. \end{aligned}$$

That the remaining terms in the series integrate to zero follows from the orthogonality property of the Legendre polynomials.

To complete the calculation we assume that F_r has been expanded in the form

$$F_r = \sum_{n=1}^{\infty} b_n(r)\,P_n(\cos\theta). \tag{29}$$

Then the Ω_n appearing in the expansion of f, see (23), may be written

$$\Omega_n(r) = \frac{1}{r}\left[\frac{d}{dr}(ra_n) + b_n\right],$$

and the integral (27) becomes

$$v_2(1) = -\frac{5}{9}\left[\int_1^\infty \frac{a_1(\alpha)+b_1(\alpha)}{\alpha}\,d\alpha - a_1(1)\right].$$

Thus we finally have
$$D_p = \tfrac{4}{3}\pi\beta\int_1^\infty \frac{a_1(\alpha)+b_1(\alpha)}{\alpha}\,d\alpha. \tag{30}$$

So far the treatment has been quite general. For the present case of a dipole field we find
$$D_p = \beta O(R_M^3). \tag{31}$$

Details of the calculation can be found in Appendix 2.

There remains the drag due to the Maxwell stress, $\mu H_i H_j - \frac{1}{2}\mu H^2\delta_{ij}$, which on a surface element of the sphere has components

$$\mu h^2\{\tfrac{1}{2}(H_r^2 - H_\theta^2),\ H_r H_\theta,\ 0\}.$$

The total contribution is a force $\rho_0 U^2 a^2 D_M$ where

$$D_M = 2\pi\beta\int_0^\pi [\tfrac{1}{2}(H_r^2 - H_\theta^2)\cos\theta - H_r H_\theta \sin\theta]\sin\theta\,d\theta. \tag{32}$$

In the present case the square bracket is

$$\tfrac{1}{2}\kappa^2(4\cos^2\theta - 5\sin^2\theta)\cos\theta + \kappa R_M[(\tfrac{1}{2}\kappa+\lambda)(3\cos^2\theta-1)^2 - 6\lambda\sin^2\theta\cos^2\theta] + O(R_M^2)$$

[see (13)], and, as we expect, the first term makes no total contribution to the drag. The remainder give
$$D_M = \beta[\tfrac{8}{5}\pi\kappa^2 R_M + O(R_M^2)]. \tag{33}$$

Note that the formulas (30) and (32) enable one to calculate the drag directly from the magnetic field. Also note that (33) is correct to quadratic terms in β and R_M since if either β or R_M is zero there can be no drag.

7. The perfectly conducting sphere

The preceding analysis applies to a sphere of arbitrary conductivity, and hence to one of arbitrarily large conductivity. However, if the conductivity is infinite, so that the magnetic field has been frozen into the sphere, the results are different. We consider in detail the dipole case again.

The field outside is again determined by the solutions $n = 1, 2$ in (10); neglecting uncorrected terms as before, we find

$$A = \exp[-\tfrac{1}{2}R_M r(1-\cos\theta)]\left(\frac{\mu'}{\mu}\right)\left\{\frac{1}{r}\sin^2\theta + \tfrac{1}{2}R_M\left(1-\frac{1}{r^2}\cos\theta\right)\sin^2\theta\right\}. \tag{34}$$

On the sphere $A = (\mu'/\mu)\sin^2\theta$, correct to $O(R_M)$, so that the normal component of magnetic induction [see (6)] has the same value as that for the dipole field frozen into the sphere. There is no requirement on the tangential component of the magnetic field; any discontinuity in it corresponds to surface currents on the sphere.

The results are now obtained without further calculation by noting that (34) can be derived from (11) by formally setting $\kappa = -2\lambda = \mu'/\mu$. Thus, according to (18), the vorticity again becomes logarithmically infinite at the sphere:

$$\omega = 8\left(\frac{\mu'}{\mu}\right)^2 R_M[1+O(R_M^2)]\sin\theta_P\log(r_P-1)$$

when P is close to the sphere. As before, there is no contribution to the drag from the pressure; the total drag (due to the Maxwell stress) is $\rho_0 U_0^2 a^2 D_M$, where [see equation (33)]

$$D_M = \frac{8\pi}{5}\left(\frac{\mu'}{\mu}\right)^2 \beta R_M[1+O(R_M)].$$

From this we see that when a sphere of high conductivity is considered to be perfectly conducting, the computed drag is in error by a factor

$$\frac{\kappa^2}{(\mu'/\mu)^2} = \left(\frac{3\mu}{2\mu+\mu'}\right)^2.$$

For example, soft iron has a μ' of the order of 300μ for weak fields, and the factor is about 10^{-4}. In equation (12), $k \doteqdot -1$ and $l \doteqdot 0$ for large μ'/μ, so that the lines of induction tend to become compressed into the sphere. On the other hand, the frozen-in field approximation would be appropriate for materials used in permanent magnets; it would also hold for a uniformly magnetized sphere.

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Appendix 1

In view of the surprising nature of this result, we feel a brief discussion of the analysis should be given here.

For ϕ_1 and ϕ_2 fixed, it is clear that the integral (17), taken over the partial ranges $(-\infty, \phi_1)$ and (ϕ_2, ϕ_p), tends to finite limits as $\psi_0 \to 0$. Hence we must show that

$$J = \int_{\phi_1}^{\phi_2} \frac{r^5 f(r,\theta;R_M)}{[(r^3-1)^2\cos^2\theta+(r^3+\frac{1}{2})^2\sin^2\theta]}d\phi_0 - I$$

also tends to a finite limit.

Now, in a sufficiently small neighbourhood, $\phi_1 \leqslant \phi_0 \leqslant \phi_2$, $0 \leqslant \psi_0 \leqslant \epsilon$, of the front stagnation point, we can find constants A and B such that

$$K = \left|\frac{r^2 f(r,\theta;R_M)}{(r^2+r+1)\cos\theta} - \frac{f(1,\pi;R_M)}{(-3)}\right| < A(r-1)+B\frac{r^3+\frac{1}{2}}{r(r^2+r+1)}\sin\theta.$$

Also, for ψ_0 fixed,

$$\frac{r^3\,d\phi_0}{(r^3-1)^2\cos^2\theta+(r^3+\frac{1}{2})^2\sin^2\theta}=\frac{dr}{(r^3-1)\cos\theta}=-\frac{r\,d\theta}{(r^3+\frac{1}{2})\sin\theta}.$$

Hence
$$|J|\leqslant\int_{\phi_0=\phi_1}^{\phi_0=\phi_2}K\frac{dr}{r-1}=A\int_{\phi_0=\phi_1}^{\phi_0=\phi_2}dr-B\int_{\phi_0=\phi_1}^{\phi_0=\phi_2}\cos\theta\,d\theta$$
$$=A(r_2-r_1)-B(\sin\theta_2-\sin\theta_1),$$

which clearly tends to a finite limit as $\psi_0\to 0$.

Appendix 2

From equations (28) and (29) we have

$$a_1(r)+b_1(r)=\frac{3}{4}\int_0^\pi F_\theta\sin^2\theta\,d\theta+\frac{3}{2}\int_0^\pi F_r\sin\theta\cos\theta\,d\theta$$
$$=\frac{3}{4}\int_0^\pi(F_\theta\sin\theta+2F_r\cos\theta)\sin\theta\,d\theta,$$

where, according to equations (15),

$$F_\theta\sin\theta+2F_r\cos\theta$$
$$=\frac{3\kappa R_M^2}{r^7}\exp\left[-R_M r(1-\cos\theta)\right]\sin^2\theta\cos\theta[(\tfrac{3}{2}\kappa r^2-\lambda)\cos^2\theta-(\tfrac{1}{2}\kappa r^2+\lambda)].\quad(35)$$

Note that the leading terms in F_θ and F_r cancel, this being a peculiarity of the dipole field.

Now, in evaluating D_p from (30), we may set $R_M=0$ in this last exponential, since this leads only to an error $O(R_M^3)$. But then (35) is antisymmetric about $\theta=\frac{1}{2}\pi$ and hence gives

$$a_1(r)+b_1(r)=0.$$

From this follows (31).

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